Motivation

Sampling Distributions of the OLS Estimators

Testing Hypotheses About a Single Population Parameter

Testing Against One-Sided Alternatives

Testing Against Two-Sided Alternatives

Testing Other Hypotheses about the β_j

Computing p-Values for t Tests

Practical (Economic) versus Statistical Significance

Confidence Intervals

Testing Multiple Exclusion

Multiple Regression Analysis - Inference

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Summer 2019

These slides were based on *Introductory Econometrics* by Jeffrey M. Wooldridge (2015)

Topics

Goal: We want to test hypothesis about the parameters β_j in the population regression model.

We want to know whether the true parameter $\beta_j =$ some value (your hypothesis).

- In order to do that, we will need to add a final assumption **MLR.6**. We will obtain the **Classical Linear Model (CLM)**

Motivation for Inference

MLR.1: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + u$
MLR.2: random sampling from the population
MLR.3: no perfect collinearity in the sample
MLR.4: $E(u|x_1, ..., x_k) = E(u) = 0$ (exogenous explanatory variables)
MLR.5: $Var(u|x_1, ..., x_k) = Var(u) = \sigma^2$ (homoskedasticity)

MLR.1 - MLR.4: Needed for unbiasedness of OLS:

$$E(\hat{\beta}_j) = \beta_j$$

MLR.1 - MLR.5: Needed to compute $Var(\hat{\beta}_j)$:

$$Var(\hat{\beta}_j) = \frac{\sigma^2}{SST_j(1 - R_j^2)}$$
$$\hat{\sigma}^2 = \frac{SSR}{(n-k-1)}$$

and for efficiency of OLS $\Rightarrow$ **BLUE**.

Topics

Sampling Distributions of the OLS Estimators

- Now we need to know the full sampling distribution of the $\hat{\beta}_j$.
- The **Gauss-Markov assumptions** don't tell us anything about these distributions.
- Based on our models, (conditional on $\{(x_{i1}, ..., x_{ik}) : i = 1, ..., n\}$) we need to have $dist(\hat{\beta}_j) = f(dist(u))$, i.e.,

$$\hat{\beta}_j \sim pdf(u)$$

- That's why we need one more assumption.

Sampling Distributions of the OLS Estimators

MRL.6 (Normality)

The population error u is independent of the explanatory variables $(x_1, ..., x_k)$ and is normally distributed with mean zero and variance σ^2:

$$u \sim Normal(0, \sigma^2)$$

Sampling Distributions of the OLS Estimators

MLR.1 - MLR.4 $\longrightarrow$ unbiasedness of OLS

Gauss-Markov assumptions: MLR.1 - MLR.4 $+$ MLR.5 (homoskedastic errors)

Classical Linear Model (CLM): Gauss-Markov $+$ MLR.6 (Normally distributed errors)

Sampling Distributions of the OLS Estimators

$$u \sim Normal(0, \sigma^2)$$

- Strongest assumption.
- **MLR.6** implies $\Rightarrow$ zero conditional mean (**MLR.4**) and homoskedasticity (**MLR.5**)
- Now we have full independence between u and $(x_1, x_2, ..., x_k)$ *(not just mean and variance independence)*
- Reason to call x_j **independent variables**.
- Recall the Normal distribution properties (see slides for **Appendix B**).

Sampling Distributions of the OLS Estimators

Figure: Distribution of u: $u \sim N(0, \sigma^2)$

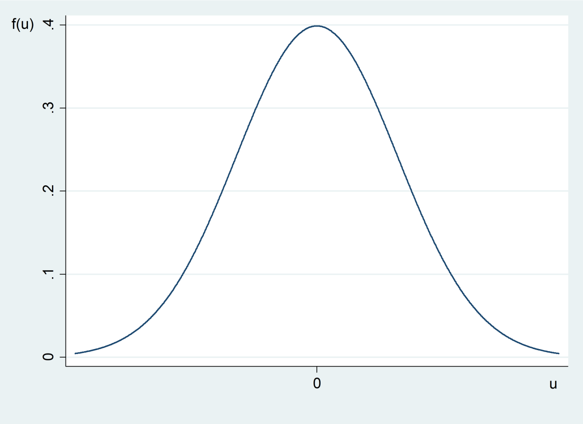
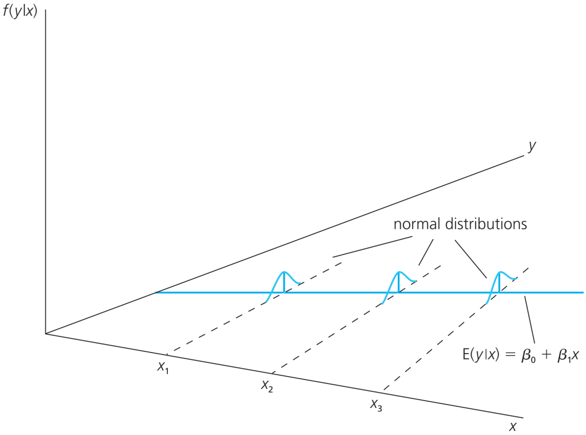


Figure: $f(y|x)$ with homoskedastic normal errors, i.e., $u \sim N(0, \sigma^2)$



Sampling Distributions of the OLS Estimators

- Property of a **Normal distribution:** if $W \sim Normal$ then $a + bW \sim Normal$ for constants a and b.

- What we are saying is that for normal r.v.s, any linear combination of them is also normally distributed.

- Because the u_i are *independent and identically distributed* (*iid*) as $Normal(0, \sigma^2)$

$$\hat{\beta}_j = \beta_j + \sum_{i=1}^{n} w_{ij} u_i \sim Normal\left(\beta_j, Var(\hat{\beta}_j)\right)$$

- Then we can apply the Central Limit Theorem.

Sampling Distributions of the OLS Estimators

Theorem: Normal Sampling Distributions

Under the **CLM** assumptions, conditional on the sample outcomes of the explanatory variables,

$$\hat{\beta}_j \sim Normal\left(\beta_j, Var(\hat{\beta}_j)\right)$$

and so

$$\frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} \sim Normal(0,1)$$

Topics

Theorem: t Distribution for Standardized Estimators

Under the **CLM** assumptions,

$$\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t_{n-k-1} = t_{df}$$

where $k+1$ is the number of unknown parameter in the population model, and $n-k-1$ is the degrees of freedom (df).

Testing Hypotheses About a Single Population Parameter: the t Test

- Compare the ratios of the **previous 2** theorems. What is the difference?
- What is the difference between $sd(\hat{\beta}_j)$ and $se(\hat{\beta}_j)$?
- Recall the t distribution properties (see slides for **Appendix B**).

- The t distribution also has a bell shape, but is more spread out than the $Normal(0,1)$.
- As $df \to \infty$,

$$t_{df} \to Normal(0,1)$$

- The difference is practically small for $df > 120$.
- See a t table.
- The next graph plots a standard normal pdf against a t_6 pdf.

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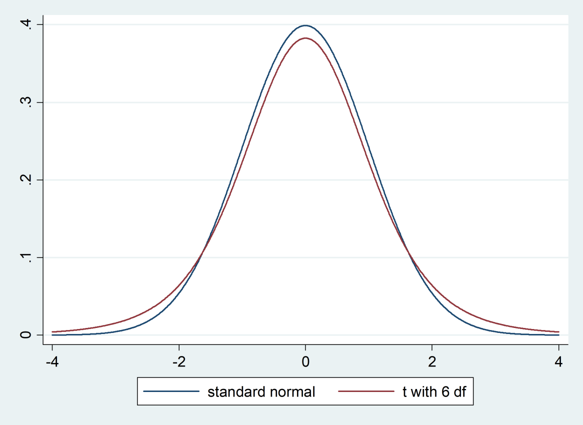
Figure: The pdfs of a standard normal and a t_6



- We use the result on the t distribution to test the null hypothesis that x_j has no partial effect on y:

$$H_0 : \beta_j = 0$$

$$\begin{aligned} lwage &= \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u \\ H_0 &: \beta_2 = 0 \end{aligned}$$

- **Interpretation of what we are doing:** Once we control for education and time on the current job ($tenure$), total workforce experience has no affect on $lwage = \log(wage)$.

Testing Hypotheses About a Single Population Parameter

- To test

$$H_0 : \beta_j = 0$$

we use the **t statistic** (or **t ratio**),

$$t_{\hat{\beta}_j} = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}$$

- In virtually all cases $\hat{\beta}_j$ is not exactly equal to zero.

- When we use $t_{\hat{\beta}_j}$, we are measuring how far $\hat{\beta}_j$ is from zero *relative to its standard error*.

Testing Against One-Sided Alternatives

- First consider the alternative

$$H_1 : \beta_j > 0$$

which means the null is effectively

$$H_0 : \beta_j \leq 0$$

- Using a positive one-sided alternative, if we reject $\beta_j = 0$, then we reject any $\beta_j < 0$, too.

- We often just state $H_0 : \beta_j = 0$ and act like we do not care about negative values.

Testing Against One-Sided Alternatives

- Because $se(\hat{\beta}_j) > 0$, $t_{\hat{\beta}_j}$ always has the same sign as $\hat{\beta}_j$.

- If the estimated coefficient $\hat{\beta}_j$ is negative, it provides no evidence against H_0 in favor of $H_1 : \beta_j > 0$.

- If $\hat{\beta}_j$ is positive, the question is: How big does $t_{\hat{\beta}_j} = \hat{\beta}_j / se(\hat{\beta}_j)$ have to be before we conclude H_0 is "unlikely"?

- Let's review the Error Types is Statistics.

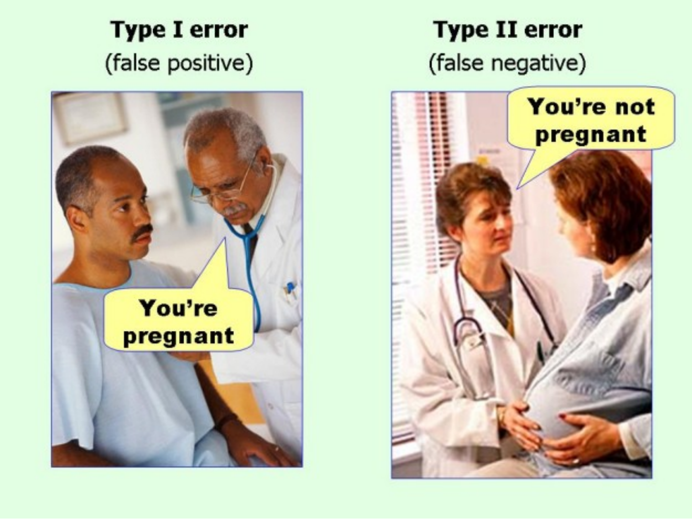
Testing Against One-Sided Alternatives

- Consider the following example:

$$H_0 \quad : \quad \text{Not pregnant}$$

$$H_1 \quad : \quad \text{Pregnant}$$

Testing Against One-Sided Alternatives



Testing Against One-Sided Alternatives

Study Finding	Reality — H0 is actually: False	Reality — H0 is actually: True
Reject H0	True Positive (Power)	False Positive Type I Error
Accept H0	False Negative Type II Error	True Negative

Testing Against One-Sided Alternatives

1. Choose a null hypothesis: $H_0 : \beta_j = 0$ (or $H_0 : \beta_j \leq 0$)

2. Choose an alternative hypothesis: $H_1 : \beta_j > 0$

3. Choose a **significance level** α **(**or simply **level**, or **size)** for the test.

   That is, the probability of rejecting the null hypothesis when it is in fact true. (Type I Error).

   Suppose we use 5%, so the probability of committing a Type I error is .05.

4. Obtain the critical value, $c > 0$, so that the **rejection rule**

$$t_{\hat{\beta}_j} > c$$

   leads to a 5% level test.

Testing Against One-Sided Alternatives

- The key is that, *under the null hypothesis*,

$$t_{\hat{\beta}_j} \sim t_{n-k-1} = t_{df}$$

and this is what we use to obtain the critical value, c.

- Suppose $df = 28$ and we use a 5% test.

- Find the **critical value** in a t-table.

table).

Testing Against One-Sided Alternatives

Numbers in each row of the table are values on a *t*-distribution with (*df*) degrees of freedom for selected right-tail (greater-than) probabilities (*p*).

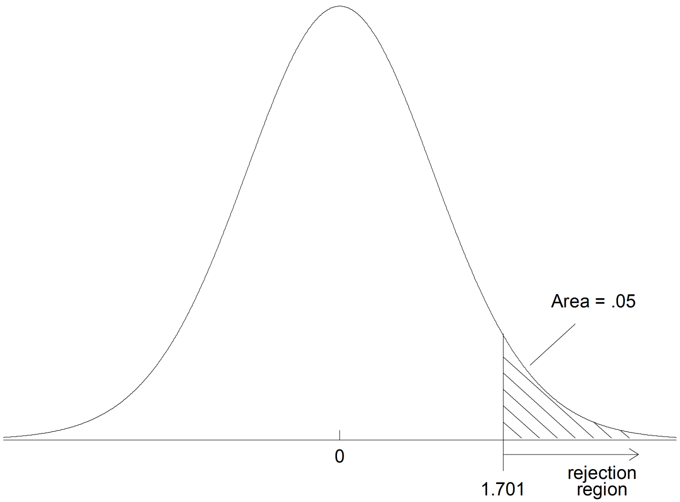
df/p	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0005
1	0.324920	1.000000	3.077684	6.313752	12.70620	31.82052	63.65674	636.6192
2	0.288675	0.816497	1.885618	2.919986	4.30265	6.96456	9.92484	31.5991
3	0.276671	0.764892	1.637744	2.353363	3.18245	4.54070	5.84091	12.9240
25	0.256060	0.684430	1.316345	1.708141	2.05954	2.48511	2.78744	3.7251
26	0.255955	0.684043	1.314972	1.705618	2.05553	2.47863	2.77871	3.7066
27	0.255858	0.683685	1.313703	1.703288	2.05183	2.47266	2.77068	3.6896
28	0.255768	0.683353	1.312527	1.701131	2.04841	2.46714	2.76326	3.6739
29	0.255684	0.683044	1.311434	1.699127	2.04523	2.46202	2.75639	3.6594
30	0.255605	0.682756	1.310415	1.697261	2.04227	2.45726	2.75000	3.6460
z	0.253347	0.674490	1.281552	1.644854	1.95996	2.32635	2.57583	3.2905
CI	———	———	80%	90%	95%	98%	99%	99.9%

Testing Against One-Sided Alternatives

- The critical value is $c = 1.701$ for 5% significance level (one-sided test).

- The following picture shows that we are conducting a **one-tailed test** (and it is these entries that should be used in the table).

Testing Against One-Sided Alternatives



- So, with $df = 28$, the rejection rule for $H_0 : \beta_j = 0$ against $H_1 : \beta_j > 0$, at the 5% level, is

$$t_{\hat{\beta}_j} > 1.701$$

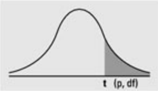
We need a t statistic greater than 1.701 to conclude there is enough evidence against H_0.

- If $t_{\hat{\beta}_j} \leq 1.701$, we fail to reject H_0 against H_1 at the 5% significance level.

Testing Against One-Sided Alternatives

- Suppose $df = 28$, but we want to carry out the test at a different significance level (often 10% level or the 1% level).

Numbers in each row of the table are values on a *t*-distribution with (*df*) degrees of freedom for selected right-tail (greater-than) probabilities (*p*).



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CI	———	———	80%	90%	95%	98%	99%	99.9%

- Thus, if $df = 28$, below are the critical values for the following significance levels: 10% level, 5% and 1% level.

$$
\begin{aligned}
c_{.10} &= 1.313 \\
c_{.05} &= 1.701 \\
c_{.01} &= 2.467
\end{aligned}
$$

Testing Against One-Sided Alternatives

If we want to reduce the probability of Type I error, we must increase the critical value (so we reject the null less often).

- If we reject at, say, the 1% level, then we must also reject at any larger level.
- If we fail to reject at, say, the 10% level – so that $t_{\hat{\beta}_j} \leq 1.313$ – then we will fail to reject at any smaller level.

Testing Against One-Sided Alternatives

- With large sample sizes – certain when $df > 120$ – we can use critical values from the standard normal distribution.

$$\begin{aligned} c_{.10} &= 1.282 \\ c_{.05} &= 1.645 \\ c_{.01} &= 2.326 \end{aligned}$$

which we can round to 1.28, 1.65, and 2.36, respectively. The value 1.65 is especially common for a one-tailed test.

Testing Against One-Sided Alternatives

- Recall our **wage** model example:

$$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_1 tenure + u$$

- First, let's label the parameters with the variable names: β_{educ}, β_{exper}, and β_{tenure}

- We would like to test:

$$H_0 : \beta_{exper} = 0$$

Interpretation: We are testing if workforce experience has no effect on a wage once education and tenure have been accounted for.

Testing Against One-Sided Alternatives

	Dependent variable:
	lwage
educ	0.092***
	(0.007)
exper	0.004**
	(0.002)
tenure	0.022***
	(0.003)
Constant	0.284***
	(0.104)
Observations	526
R2	0.316
Adjusted R2	0.312
Residual Std. Error	0.441 (df = 522)
F Statistic	80.391*** (df = 3; 522)
Note:	*p<0.1; **p<0.05; ***p<0.01

Testing Against One-Sided Alternatives

- What is the t_{exper}?

$$t_{exper} = \frac{0.004}{0.002} = 2.00$$

- Now what do you do with this number?
- How many df do we have?
- Which table could I use?
- Using a standard normal table: the one-sided critical value at the 5% level, 1.645.

Statistical Significance X Economic Importance/Interpretation

• So "$\hat{\beta}_{exper}$ is **statistically significant**" at 5% level significance level (one-sided test).

• The estimated effect of $exper$, which is its **economic importance** should be interpreted as: another year of experience, holding $educ$ and $tenure$ fixed, is estimated to be worth about 0.4%.

Testing Against One-Sided Alternatives

- For the negative one-sided alternative,

$$H_0 : \beta_j \geq 0$$

$$H_1 : \beta_j < 0$$

we use a symmetric rule. But the rejection rule is

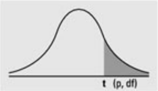
$$t_{\hat{\beta}_j} < -c$$

where c is chosen in the same way as in the positive case.

Testing Against One-Sided Alternatives

- With $df = 28$ and we want to test at a 5% significance level, what is the critical value?

Numbers in each row of the table are values on a *t*-distribution with (*df*) degrees of freedom for selected right-tail (greater-than) probabilities (*p*).



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CI	———	———	80%	90%	95%	98%	99%	99.9%

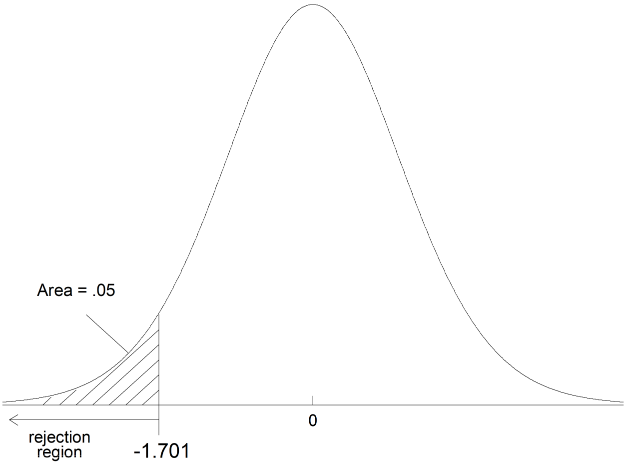
Testing Against One-Sided Alternatives

Intuition: We must see a significantly negative value for the t statistic to reject the null hypothesis in favor of the alternative hypothesis.

- With $df = 28$ and a 5% test, the critical value is $c = -1.701$, so the rejection rule is

$$t_{\hat{\beta}_j} < -1.701$$

Testing Against One-Sided Alternatives



Reminder about Testing

- Our hypotheses involve the unknown population values, β_j.

- If in a our set of data we obtain, say, $\hat{\beta}_j = 2.75$, we do not write the null hypothesis as

$$H_0 : 2.75 = 0$$

(which is obviously false).

Testing Against One-Sided Alternatives

- Nor do we write

$$H_0 : \hat{\beta}_j = 0$$

(which is also false except in the very rare case that our estimate is exactly zero).

- We do not test hypotheses about the estimate! We know what it is once we collect the sample. We hypothesize about the unknown population value, β_j.

Testing Against Two-Sided Alternatives

- Sometimes we do not know ahead of time whether a variable definitely has a positive effect or a negative effect.
- So, in this case the hypothesis should be written as:

$$\begin{aligned} H_0 &: \beta_j = 0 \\ H_1 &: \beta_j \neq 0 \end{aligned}$$

- Testing against the **two-sided alternative** is usually the default. It prevents us from looking at the regression results and then deciding on the alternative.

Testing Against Two-Sided Alternatives

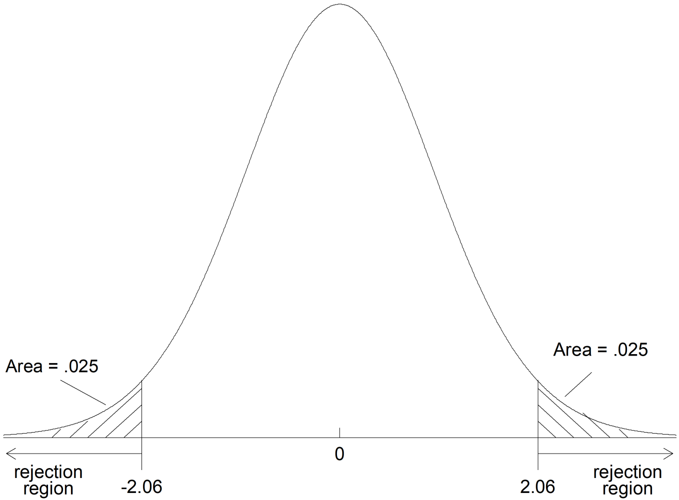
- Now we reject if $\hat{\beta}_j$ is sufficiently large in magnitude, either positive or negative. We again use the t statistic $t_{\hat{\beta}_j} = \hat{\beta}_j / se(\hat{\beta}_j)$, but now the rejection rule is

Two-tailed test

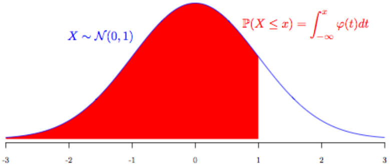
$$\left| t_{\hat{\beta}_j} \right| > c$$

- For example, if we use a 5% level test and $df = 25$, the two-tailed cv is 2.06. The two-tailed cv is, in this case, the 97.5 percentile in the t_{25} distribution. (Compare the one-tailed cv, about 1.71, the 95^{th} percentile in the t_{25} distribution).

Testing Against Two-Sided Alternatives



Testing Against Two-Sided Alternatives



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Testing Against One-Sided Alternatives

	Dependent variable:
	lwage
educ	0.092***
	(0.007)
exper	0.004**
	(0.002)
tenure	0.022***
	(0.003)
Constant	0.284***
	(0.104)
Observations	526
R2	0.316
Adjusted R2	0.312
Residual Std. Error	0.441 (df = 522)
F Statistic	80.391*** (df = 3; 522)
Note:	*p<0.1; **p<0.05; ***p<0.01

Testing Against Two-Sided Alternatives

- When we reject $H_0 : \beta_j = 0$ against $H_1 : \beta_j \neq 0$, we often say that $\hat{\beta}_j$ is **statistically different from zero** and usually mention a significance level.

As in the one-sided case, we also say $\hat{\beta}_j$ is **statistically significant** when we can reject $H_0 : \beta_j = 0$.

Testing Other Hypotheses about the β_j

- Testing the null $H_0 : \beta_j = 0$ is the standard practice.

- **R**, Stata, EViews and all the other regression packages automatically report the t statistic for **this hypothesis** (i.e., two-sided test).

Testing Other Hypotheses about the β_j

- What if we want to test a different null value? For example, in a constant-elasticity consumption function,

$$\log(cons) = \beta_0 + \beta_1 \log(inc) + \beta_2 famsize + \beta_3 pareduc + u$$

we might want to test

$$H_0 : \beta_1 = 1$$

which means an income elasticity equal to one. (We can be pretty sure that $\beta_1 > 0$.)

Testing Other Hypotheses about the β_j

Important observation

$$t_{\hat{\beta}_j} = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}$$

is *only* for $H_0 : \beta_j = 0$.

Testing Other Hypotheses about the β_j

- More generally, suppose the null is

$$H_0 : \beta_j = a_j$$

where we specify the value a_j

- It is easy to extend the t statistic:

$$t = \frac{(\hat{\beta}_j - a_j)}{se(\hat{\beta}_j)}$$

The t statistic just measures how far our estimate, $\hat{\beta}_j$, is from the hypothesized value, a_j, relative to $se(\hat{\beta}_j)$.

Testing Other Hypotheses about the β_j

General expression for general t testing

$$t = \frac{(\textit{estimate} - \textit{hypothesized value})}{\textit{standard error}}$$

- The alternative can be one-sided or two-sided.
- We choose critical values in exactly the same way as before.

Testing Other Hypotheses about the β_j

- The language needs to be suitably modified. If, for example,

$$\begin{aligned} H_0 &: \beta_j = 1 \\ H_1 &: \beta_j \neq 1 \end{aligned}$$

is rejected at the 5% level, we say "$\hat{\beta}_j$ is statistically different from one at the 5% level." Otherwise, $\hat{\beta}_j$ is "not statistically different from one." If the alternative is $H_1 : \beta_j > 1$, then "$\hat{\beta}_j$ is statistically greater than one at the 5% level."

Testing Other Hypotheses about the β_j

Example: Crime, police officers and enrollment on college campuses
Let's do the following hypothesis test:

$$\begin{aligned} \log(crime) &= \beta_0 + \beta_1 police + \beta_2 log(enroll) + u \\ H_0 &: \beta_1 = 1 \\ H_1 &: \beta_1 > 1 \end{aligned}$$

Testing Other Hypotheses about the β_j

	Dependent variable:
	log(crime)
police	0.0240***
	(0.0073)
log(enroll)	0.9767***
	(0.1373)
Constant	-4.3758***
	(1.1990)
Observations	97
R2	0.6277
Adjusted R2	0.6198
Residual Std. Error	0.8516 (df = 94)
F Statistic	79.2389*** (df = 2; 94)
Note:	*p<0.1; **p<0.05; ***p<0.01

Computing p-Values for t Tests

- The traditional approach to testing, where we choose a significance level ahead of time, has a component of **arbitrariness**.
- Different researchers prefer different significance levels (10%, 5%, 1%).
- Committing to a significance level ahead of time can hide useful information about the outcome of hypothesis test.
- **Example:** (On white board)

- Rather than have to specify a level ahead of time, or discuss different traditional significance levels (10%, 5%, 1%), it is better to answer the following question:

Intuition: Given the observed value of the t statistic, what is the *smallest* significance level at which I can reject H_0?

- The smallest level at which the null can be rejected is known as the p-**value** of a test.

Computing p-Values for t Tests

p-value

For t testing against a two-sided alternative,

$$p\text{-value} = P(|T| > |t|)$$

where t is the value of the t statistic and T is a random variable with the t_{df} distribution.

- The p-value is a probability, so it is between zero and one.

Computing p-Values for t Tests

One way to think about the p-values is that it uses the observed statistic as the critical value, and then finds the significance level of the test using that critical value.

- Usually we just report p-values for two-sided alternatives.

Computing p-Values for t Tests

Mnemonic Device

Small p-values are evidence *against* the null hypothesis.

Large p-values provide little evidence *against* the null hypothesis.

Intuition: p-value is the probability of observing a statistic as extreme as we did if the null hypothesis is true.

Computing p-Values for t Tests

- If p-value $= .50$, then there is a 50% chance of observing a t as large as we did (in absolute value). This is not enough evidence against H_0.

- If p-value $= .001$, then the chance of seeing a t statistic as extreme as we did is $.1\%$.

- We can conclude that we got a very rare sample (*unlikely!*) or that the null hypothesis is very likely false.

Computing p-Values for t Tests

- From

$$p\text{-value} = P(|T| > |t|)$$

we see that as $|t|$ increases the p-value decreases.

Large absolute t statistics are **associated** with small p-values.

Example:

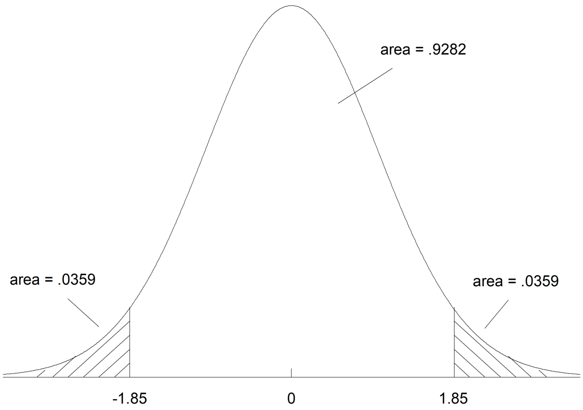
- Suppose $df = 40$ and, from our data, we obtain $t = 1.85$ or $t = -1.85$. Then

$$p\text{-value} = P(|T| > 1.85) = 2P(T > 1.85) = 2(.0359) = .0718$$

where $T \sim t_{40}$.

Computing p-Values for t Tests

Figure: t distribution with 40 degrees of freedom



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Computing p-Values for t Tests

- Given p-value, we can carry out a test at any significance level.

If α is the chosen level, then

Reject H_0 if p-value $< \alpha$

Example

Suppose we obtained p-value $= .0718$. This means that we reject H_0 at the 10% level but not the 5% level. We reject at 8% but not at 7%.

Practical versus Statistical Significance

- t testing is purely about *statistical significance*.

- It does not directly speak to the issue of whether a variable has a practically, or economically large effect.

Practical (Economic) Significance depends on the size (and sign) of $\hat{\beta}_j$.

Statistical Significance depends on $t_{\hat{\beta}_j}$.

Practical (Economic) versus Statistical Significance

It is possible estimate **practically large effects** but have the estimates so imprecise that they are **statistically insignificant**.

Common with small data sets (but not only small data sets).

X

It is possible to get estimates that are **statistically significant** (often with very small p-values) but are **not practically large**.

Common with very large data sets.

Topics

- **Under the CLM assumptions**, rather than just testing hypotheses about parameters it is also useful to construct **confidence intervals** (also know as **interval estimates**).

Intuition: If you could obtain several random samples data, the **confidence interval** tells you that, for a 95% CI, your true β_j will lie in this interval $[\beta_j^{lower}, \beta_j^{upper}]$ for 95% of the samples.

Confidence Intervals

- We will construct CIs of the form

$$\hat{\beta}_j \pm c \cdot se(\hat{\beta}_j)$$

where $c > 0$ is chosen based on the **confidence level**.

- We will use a 95% confidence level, in which case c comes from the 97.5 percentile in the t_{df} distribution.

- Therefore, c is the 5% critical value against a two-sided alternative.

Confidence Intervals

Rule of Thumb

- For, $df \geq 120$, an approximate 95% CI is:

$$\hat{\beta}_j \pm 2 \cdot se(\hat{\beta}_j) \text{ or } \left[\hat{\beta}_j - 2 \cdot se(\hat{\beta}_j), \hat{\beta}_j + 2 \cdot se(\hat{\beta}_j)\right]$$

- For small df, the exact percentiles should be obtained from a t table.

Confidence Intervals

Find the 95% CI for the parameters from the following regression:

	Dependent variable:
	lwage
educ	0.092***
	(0.007)
exper	0.004**
	(0.002)
tenure	0.022***
	(0.003)
Constant	0.284***
	(0.104)
Observations	526
R2	0.316
Adjusted R2	0.312
Residual Std. Error	0.441 (df = 522)
F Statistic	80.391*** (df = 3; 522)
Note:	*p<0.1; **p<0.05; ***p<0.01

Confidence Intervals

- The correct way to interpret a CI is to remember that the endpoints, $\hat{\beta}_j - c \cdot se(\hat{\beta}_j)$ and $\hat{\beta}_j + c \cdot se(\hat{\beta}_j)$, **change** with each sample (or at least can change).

Endpoints are random outcomes that depend on the data we draw.

Confidence Intervals

A 95% CI means is that for 95% of the random samples that we draw from the population,

the interval we compute using the rule $\hat{\beta}_j \pm c \cdot se(\hat{\beta}_j)$

will include the value β_j.

But for a particular sample we do not know whether β_j is in the interval.

- This is similar to the idea that unbiasedness of $\hat{\beta}_j$ does *not* means that $\hat{\beta}_j = \beta_j$. Most of the time $\hat{\beta}_j$ is not β_j. Unbiasedness means $E(\hat{\beta}_j) = \beta_j$.

Topics

Testing Multiple Exclusion Restrictions

- Sometimes want to test **more than one hypothesis**, which then includes **multiple parameters**.
- Generally, it is not valid to look at individual t statistics.
- We need a specific statistic used to test **joint hypotheses**.

Testing Multiple Exclusion Restrictions

Example:

$$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u$$

- Let's consider the following null hypothesis:

$$H_0 : \beta_2 = 0, \ \beta_3 = 0$$

- **Exclusion Restrictions**: We want to know if we can exclude some variables jointly.

Testing Multiple Exclusion Restrictions

- To test H_0, we need a **joint (multiple) hypotheses test**.

- A t statistic can be used for a single exclusion restriction; it does not take a stand on the values of the other parameters.

- We are considering the alternative to be:

$$H_1 : H_0 \text{ is not true}$$

- So, H_1 means **at least one** of betas is different from zero.

Testing Multiple Exclusion Restrictions

- The original model, containing all variables, is the **unrestricted model**:

$$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u$$

- When we impose $H_0 : \beta_2 = 0, \ \beta_3 = 0$, we get the **restricted model**:

$$\log(wage) = \beta_0 + \beta_1 educ + u$$

Testing Multiple Exclusion Restrictions

- We want to see how the fit deteriorates as we remove the two variables.
- We use, initially, the **sum of squared residuals** from the two regressions.
- It is an algebraic fact that the SSR must increase (or, at least not fall) when explanatory variables are dropped. So,

$$SSR_r \geq SSR_{ur}$$

F test

Does the SSR increase proportionately by enough to conclude the restrictions under H_0 are false?

Testing Multiple Exclusion Restrictions

- In the general model:

$$y = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k + u$$

we want to test that the last q variables can be excluded:

$$H_0 : \beta_{k-q+1} = 0, ..., \beta_k = 0$$

- We get SSR_{ur} from estimating the full model.

Testing Multiple Exclusion Restrictions

- The restricted model we estimate to get SSR_r drops the last q variables (q exclusion restrictions):

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_{k-q} x_{k-q} + u$$

- The **F statistic** uses a degrees of freedom adjustment. In general, we have

$$F = \frac{(SSR_r - SSR_{ur})/(df_r - df_{ur})}{SSR_{ur}/df_{ur}} = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}$$

where q is the number of exclusion restrictions imposed under the null ($q = 2$ in our example).

$$\begin{aligned} q &= \textbf{numerator df} = df_r - df_{ur} \\ n-k-1 &= \textbf{denominator df} = df_{ur} \end{aligned}$$

- The denominator of the F statistic, SSR_{ur}/df_{ur}, is the unbiased estimator of σ^2 from the unrestricted model.
- Note that $F \geq 0$, and $F > 0$ virtually always holds.
- As a computational device, sometimes the formula

$$F = \frac{(SSR_r - SSR_{ur})}{SSR_{ur}} \cdot \frac{(n-k-1)}{q}$$

is useful.

Testing Multiple Exclusion Restrictions

- Using classical testing, the rejection rule is of the form

$$F > c$$

where c is an appropriately chosen **critical value**.

Distribution of F statistic

Under H_0 (the q exclusion restrictions)

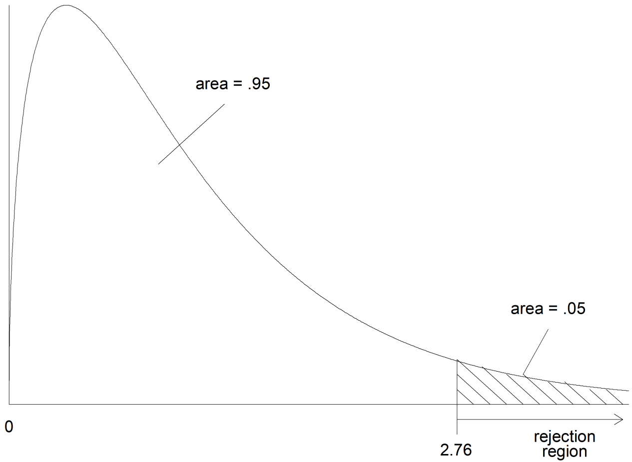
$$F \sim F_{q,n-k-1}$$

i.e., it has an F distribution with $(q, n-k-1)$ degrees of freedom.

- Recall the F distribution (see slides for **Appendix B**).

Testing Multiple Exclusion Restrictions

- Suppose $q = 3$ and $n - k - 1 = df_{ur} = 60$. Then the 5% cv is 2.76.



R-Squared Form of the F Statistic

Question: Is there a way to compute the F statistic with the information reported in the standard output from any econometric/statistcal package?

- The R-squared is always reported.
- The SSR is not reported most of the time.
- It turns out that F tests for exclusion restrictions can be computed entirely from the R-squareds for the restricted and unrestricted models.
- Notice that,

$$\begin{aligned} SSR_r &= (1 - R_r^2)SST \\ SSR_{ur} &= (1 - R_{ur}^2)SST \end{aligned}$$

R-Squared Form of the F Statistic

- Therefore,

$$F = \frac{(R^2_{ur} - R^2_r)/q}{(1 - R^2_{ur})/(n-k-1)}$$

- Notice how R^2_{ur} comes first in the numerator.
- We know $R^2_{ur} \geq R^2_r$ so this ensures $F \geq 0$.

R-Squared Form of the F Statistic

Example

unrestricted model: $\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u$

restricted model: $\log(wage) = \beta_0 + \beta_1 educ + u$

R-Squared Form of the F Statistic

	Dependent variable: lwage
educ	0.092***
	(0.007)
exper	0.004**
	(0.002)
tenure	0.022***
	(0.003)
Constant	0.284***
	(0.104)
Observations	526
R2	0.316
Adjusted R2	0.312
Residual Std. Error	0.441 (df = 522)
F Statistic	80.391*** (df = 3; 522)
Note:	*p<0.1; **p<0.05; ***p<0.01

	Dependent variable: lwage
educ	0.083***
	(0.008)
Constant	0.584***
	(0.097)
Observations	526
R2	0.186
Adjusted R2	0.184
Residual Std. Error	0.480 (df = 524)
F Statistic	119.582*** (df = 1; 524)
Note:	*p<0.1; **p<0.05; ***p<0.01

R-Squared Form of the F Statistic

- We say that $exper$, and $tenure$ are **jointly statistically significant** (or just **jointly significant**), in this case, at any small significance level we want.

- The F statistic does not allow us to tell which of the population coefficients are different from zero. And the t statistics do not help much in this example.

The F Statistic for Overall Significance of a Regression

- The F statistic in the **R** output tests a very special null hypothesis.
- In the model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + 0$$

the null is that **all slope coefficients are zero**, i.e,

$$H_0 : \beta_1 = 0, \beta_2 = 0, ..., \beta_k = 0$$

- This means that none of the x_j helps explain y.
- If we cannot reject this null, we have found no factors that explain y.

The F Statistic for Overall Significance of a Regression

- For this test,

$R_r^2 = 0$ (no explanatory variables under H_0).
$R_{ur}^2 = R^2$ from the regression.

$$F = \frac{R^2/k}{(1-R^2)/(n-k-1)} = \frac{R^2}{(1-R^2)} \cdot \frac{(n-k-1)}{k}$$

The F Statistic for Overall Significance of a Regression

- As R^2 increases, so does F.
- A small R^2 can lead F to be significant.
- If the $df = n - k - 1$ is large (because of large n), F can be large even with a "small" R^2.
- Increasing k decreases F.

KU

The F Statistic for Overall Significance of a Regression

	Dependent variable:
	lwage
educ	0.092***
	(0.007)
exper	0.004**
	(0.002)
tenure	0.022***
	(0.003)
Constant	0.284***
	(0.104)
Observations	526
R2	0.316
Adjusted R2	0.312
Residual Std. Error	0.441 (df = 522)
F Statistic	80.391*** (df = 3; 522)
Note:	*p<0.1; **p<0.05; ***p<0.01

	Dependent variable:
	lwage
Constant	1.623***
	(0.023)
Observations	526
R2	0.000
Adjusted R2	0.000
Residual Std. Error	0.532 (df = 525)
Note:	*p<0.1; **p<0.05; ***p<0.01