

The Simple Regression Model

Caio Vigo

The University of Kansas
Department of Economics

Fall 2019

These slides were based on *Introductory Econometrics* by Jeffrey M. Wooldridge (2015)

Definition of the Simple Regression Model

Deriving the Ordinary Least Squares Estimates

Properties of OLS on any Sample of Data

Units of Measurement and Functional Form

Using the Natural Logarithm in Simple Regression

Expected Value of OLS

- ① Definition of the Simple Regression Model
- ② Deriving the Ordinary Least Squares Estimates
- ③ Properties of OLS on any Sample of Data
- ④ Units of Measurement and Functional Form
Using the Natural Logarithm in Simple Regression
- ⑤ Expected Value of OLS

- **What type of analysis will we do?** Cross-sectional analysis
- **First step:** Clearly define what is your population (in what you are interested to study).
- **Second Step:** There are two variables, x and y , and we would like to “study how y varies with changes in x .”
- **Third Step:** We assume we can collect a random sample from the population of interest.

Now we will learn to write our first econometric model, derive an estimator (**what’s an estimator again?**) and use this estimator in our sample.

We must confront three issues:

- ① How do we allow factors other than x to affect y ? There is never an exact relationship between two variables.
- ② What is the functional relationship between y and x ?
- ③ How can we be sure we are capturing a *ceteris paribus* relationship between y and x ?

Consider the following equation relating y to x :

$$y = \beta_0 + \beta_1 x + u,$$

which is assumed to hold in the population of interest.

- This equation defines the **simple linear regression model** (or *two-variable regression model*, or *bivariate linear regression model*).

- y and x are not treated symmetrically. We want to explain y in terms of x .

x explains y

$x \longrightarrow y$

- **Example:**

size of the city x , **explains** number of crimes (y) (**not the other way around**).

Definition of the Simple Regression Model

Deriving the Ordinary Least Squares Estimates

Properties of OLS on any Sample of Data

Units of Measurement and Functional Form

Using the Natural Logarithm in Simple Regression

Expected Value of OLS

y	x
Dependent Variable	Independent Variable
Explained Variable	Explanatory Variable
Response Variable	Control Variable
Predicted Variable	Predictor Variable
Regressand	Regressor

$$y = \beta_0 + \beta_1 x + u$$

This equation explicitly allows for other factors, contained in u , to affect y .

This equation also addresses the functional form issue (in a simple way).

Namely, y is assumed to be *linearly* related to x .

We call β_0 the **intercept parameter** and β_1 the **slope parameter**. These describe a population, and our ultimate goal is to estimate them.

The simple linear regression model equation

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

- The equation also addresses the *ceteris paribus* issue. In

$$y = \beta_0 + \beta_1 x + u,$$

all other factors that affect y are in u . We want to know how y changes when x changes, *holding u fixed*.

- Let Δ denote “change.” Then holding u fixed means $\Delta u = 0$. So

$$\begin{aligned} \Delta y &= \beta_1 \Delta x + \Delta u \\ &= \beta_1 \Delta x \quad \text{when } \Delta u = 0. \end{aligned}$$

- This equation effectively defines β_1 as a slope, with the only difference being the restriction $\Delta u = 0$.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Example: Yield and Fertilizer

- A model to explain crop yield to fertilizer use is

$$yield = \beta_0 + \beta_1 fertilizer + u,$$

where u contains land quality, rainfall on a plot of land, and so on. The slope parameter, β_1 , is of primary interest: it tells us how *yield* changes when the amount of fertilizer changes, holding all else fixed.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Example: Wage and Education

$$wage = \beta_0 + \beta_1 educ + u$$

where u contains somewhat nebulous factors (“ability”) but also past workforce experience and tenure on the current job.

$$\Delta wage = \beta_1 \Delta educ \quad \text{when } \Delta u = 0$$

The simple linear regression model equation

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

We said we must confront three issues:

1. How do we allow factors other than x to affect y ?

Answer: u

2. What is the functional relationship between y and x ?

Answer: Linear (x has a linear effect on y)

3. How can we be sure we are capturing a ceteris paribus relationship between y and x ?

Answer: Related with $\Delta u = 0$

- We have argued that the simple regression model

$$y = \beta_0 + \beta_1 x + u$$

addresses each of them.

To estimate β_1 and β_0 from a random sample we also need to **restrict how u and x are related to each other.**

- Recall that x and u are properly viewed as having distributions in the population.
- What we must do is restrict the way in which u and x relate to each other in the **population.**
- First, we make a simplifying assumption that is without loss of generality: the average, or expected, value of u is zero in the population:

$$E(u) = 0$$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

- Normalizing u should cause no impact in the most important parameter: β_1
- The presence of β_0 in

$$y = \beta_0 + \beta_1 x + u$$

allows us to assume $E(u) = 0$.

- If the average of u is different from zero, we just adjust the intercept, leaving the slope the same. If $\alpha_0 = E(u)$ then we can write

$$y = (\beta_0 + \alpha_0) + \beta_1 x + (u - \alpha_0),$$

where the new error, $u - \alpha_0$, has a zero mean.

We need to restrict the dependence between u and x

- **Option 1:** Uncorrelated

We could assume u and x **uncorrelated** in the population:

$$\text{Corr}(x, u) = 0$$

It implies **only** that u and x are not **linearly** related. **Not good enough.**

- **Option 2:** Mean independence

The mean of the error (i.e., the mean of the unobservables) is the same across all slices of the population determined by values of x .

We represent it by:

$$E(u|x) = E(u), \text{ all values } x,$$

And we say that u is **mean independent** of x

- Suppose u is “ability” and x is years of education. We need, for example,

$$E(\textit{ability}|x = 8) = E(\textit{ability}|x = 12) = E(\textit{ability}|x = 16)$$

so that the average ability is the same in the different portions of the population with an 8th grade education, a 12th grade education, and a four-year college education.

- Combining $E(u|x) = E(u)$ (the substantive assumption) with $E(u) = 0$ (a normalization) gives

$$E(u|x) = 0, \text{ all values } x$$

- Called the **zero conditional mean assumption**.

- First, recall the properties of conditional expectation. (*see slides with a review of Probability*)
- Now, take the conditional expectation of our *Simple Linear Regression Function*. Then, we get:

$$\begin{aligned}
 E(y|x) &= \beta_0 + \beta_1 x + E(u|x) \\
 &= \beta_0 + \beta_1 x
 \end{aligned}$$

which shows the **population regression function** is a linear function of x .

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Figure: Example: The goal is to explain **weekly consumption expenditure** in terms of **weekly income**

$Y \downarrow$ / $X \rightarrow$	80	100	120	140	160	180	200	220	240	260
Weekly family consumption expenditure Y, \$	55	65	79	80	102	110	120	135	137	150
	60	70	84	93	107	115	136	137	145	152
	65	74	90	95	110	120	140	140	155	175
	70	80	94	103	116	130	144	152	165	178
	75	85	98	108	118	135	145	157	175	180
	-	88	-	113	125	140	-	160	189	185
	-	-	-	115	-	-	-	162	-	191
Total	325	462	445	707	678	750	685	1043	966	1211
Conditional means of Y, $E(Y X)$	65	77	89	101	113	125	137	149	161	173

Source: Gujarati, Damodar (2002). Basic Econometrics.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

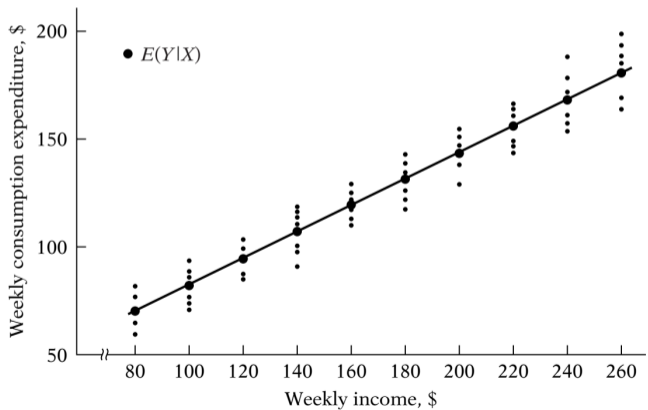
Figure: Conditional Probabilities of the data

$p(Y X_i)$ ↓ $X \rightarrow$	80	100	120	140	160	180	200	220	240	260
Conditional probabilities $p(Y X_i)$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{7}$
	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{7}$
	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{7}$
	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{7}$
	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{7}$
	—	$\frac{1}{6}$	—	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{6}$	—	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{7}$
	—	—	—	$\frac{1}{7}$	—	—	—	$\frac{1}{7}$	—	$\frac{1}{7}$
Conditional means of Y	65	77	89	101	113	125	137	149	161	173

Source: Gujarati, Damodar (2002). Basic Econometrics.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

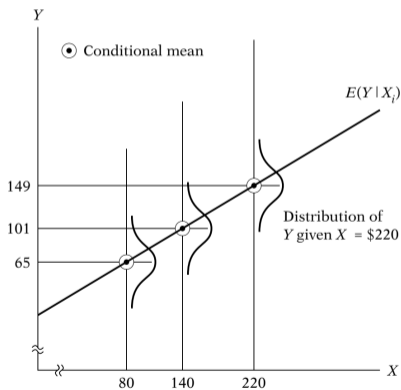
Figure: Conditional distribution of expenditure for various levels of income



Source: Gujarati, Damodar (2002). Basic Econometrics.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Figure: The Population Regression Function (PRF)



Source: Gujarati, Damodar (2002). Basic Econometrics.

- The straight line in the previous graph is the PRF, $E(y|x) = \beta_0 + \beta_1 x$. The conditional distribution of y at three different values of x are superimposed.
- For a given value of x , we see a range of y values: remember, $y = \beta_0 + \beta_1 x + u$, and u has a distribution in the population.
- In practice, we never know the **population intercept and slope**.

- Assuming we know the PRF, consider this example:

Example

- Suppose for the population of students attending a university, we know the PRF:

$$E(\text{colGPA} | \text{hsGPA}) = 1.5 + 0.5 \text{hsGPA},$$

- For this example, what is y ? what is x ? What is the slope? What's the intercept?
- If $\text{hsGPA} = 3.6$ what's the expected college GPA? $1.5 + 0.5(3.6) = 3.3$

Definition of the Simple Regression Model

Deriving the Ordinary Least Squares Estimates

Properties of OLS on any Sample of Data

Units of Measurement and Functional Form

Using the Natural Logarithm in Simple Regression

Expected Value of OLS

- ① Definition of the Simple Regression Model
- ② Deriving the Ordinary Least Squares Estimates
- ③ Properties of OLS on any Sample of Data
- ④ Units of Measurement and Functional Form
Using the Natural Logarithm in Simple Regression
- ⑤ Expected Value of OLS

- Given data on x and y , how can we estimate the population parameters, β_0 and β_1 ?
- Let $\{(x_i, y_i) : i = 1, 2, \dots, n\}$ be a **random sample** of size n (the number of observations) from the population. Think of this as a random sample.

Derivation: (On white board)

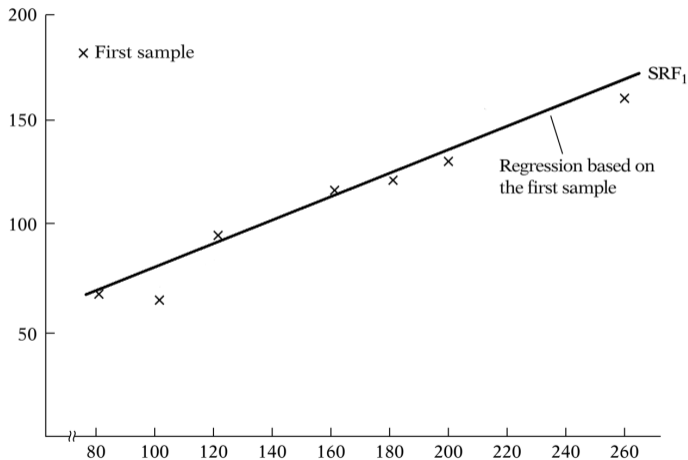
Estimator for β_0

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

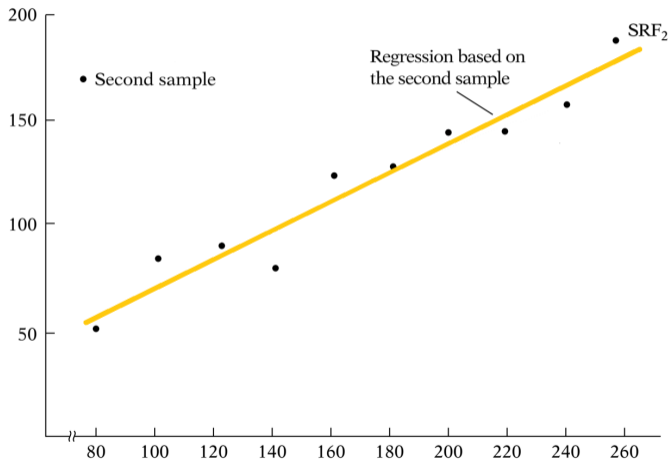
Estimator for β_1

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\text{Sample Covariance}(x, y)}{\text{Sample Variance}(x)} \\ &= \frac{S_{x,y}}{S_x^2} \\ &= \hat{\rho}_{x,y} \frac{\hat{\sigma}_y}{\hat{\sigma}_x} \end{aligned}$$

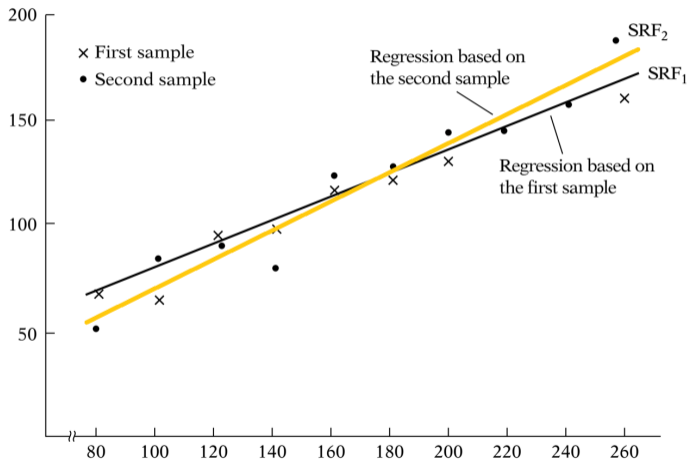
- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates**
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS



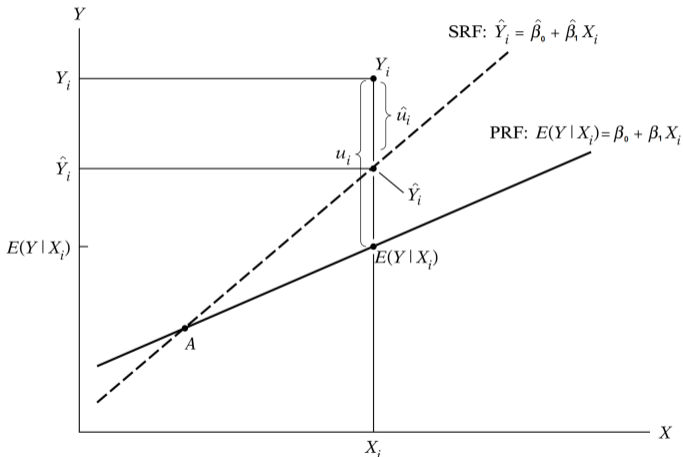
- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates**
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS



- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates**
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS



- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS



Source: Gujarati, Damodar (2002). Basic Econometrics.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Example: Effects of Education on Hourly Wage

- Data: random sample from the US workforce population in 1976.
wage: dollars per hour,
educ: highest grade completed (years of education).

- The estimated equation is

$$\widehat{wage} = -0.90 + 0.54 \text{ educ}$$

$$n = 526$$

- Each additional year of schooling is estimated to be worth \$0.54.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

The function

$$\widehat{wage} = -0.90 + 0.54 educ$$

is the **OLS (or sample) regression line.**

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

```
> stargazer(regression_wage1, type='text', align=TRUE, digits=2)
```

```
=====
                        Dependent variable:
-----
                        wage
-----
educ                    0.54***
                        (0.05)

Constant                -0.90
                        (0.68)

-----
Observations            526
R2                      0.16
Adjusted R2             0.16
Residual Std. Error    3.38 (df = 524)
F Statistic             103.36*** (df = 1; 524)
=====
Note:                    *p<0.1; **p<0.05; ***p<0.01
```

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

- When we write the simple linear regression model,

$$wage = \beta_0 + \beta_1 educ + u,$$

it applies to the population, so we do not know β_0 and β_1 .

- $\hat{\beta}_0 = -0.90$ and $\hat{\beta}_1 = 0.54$ are our *estimates* from this particular sample.
- These estimates may or may not be close to the population values. If we obtain another sample, the estimates would almost certainly change.

- If $educ = 0$, the predicted wage is:

$$\widehat{wage} = -0.90 + 0.54(0) = -0.90$$

The predicted value does not fit in reality.

Mainly because when we extrapolate outside the majority range of our data can produce strange predictions.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

- When $educ = 8$, the predicted wage is:

$$\widehat{wage} = -0.90 + 0.54(8) = 3.42$$

which we can think of as our estimate of the average wage in the population when $educ = 8$.

Sample Regression Line (SRF)

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \quad i = 1, \dots, n$$

Also known as:

- OLS Regression Line
- Sample Regression Function
- OLS Regression Function
- Estimated Equation

Population Regression Function (PRF)

Since the simple linear regression model (or just econometric model) is:

$$y_i = \beta_0 + \beta_1 x_i + u$$

Then, the PRF is:

$$\Rightarrow E(y_i|x) = \beta_0 + \beta_1 x_i \quad i = 1, 2, \dots, n$$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Residuals

$$\hat{u}_i = y_i - \hat{y}_i \quad i = 1, 2, \dots, n$$

Error Term

$$\begin{aligned} u_i &= y_i - E(y|x) \\ &= y_i - \beta_0 - \beta_1 x_i \quad i = 1, 2, \dots, n \end{aligned}$$

Definition of the Simple Regression Model

Deriving the Ordinary Least Squares Estimates

Properties of OLS on any Sample of Data

Units of Measurement and Functional Form

Using the Natural Logarithm in Simple Regression

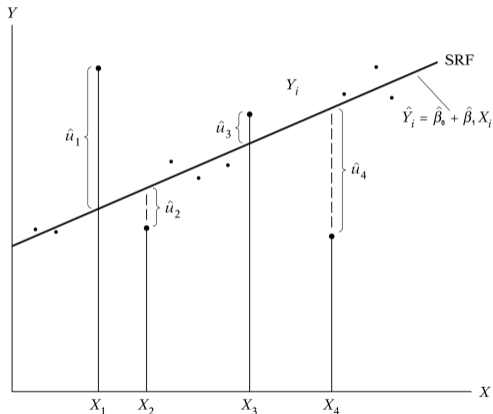
Expected Value of OLS

- ① Definition of the Simple Regression Model
- ② Deriving the Ordinary Least Squares Estimates
- ③ Properties of OLS on any Sample of Data
- ④ Units of Measurement and Functional Form
Using the Natural Logarithm in Simple Regression
- ⑤ Expected Value of OLS

- Recall that the OLS residuals are

$$\hat{u}_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \quad , \quad i = 1, 2, \dots, n$$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data**
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS



Source: Gujarati, Damodar (2002). Basic Econometrics.

- Some residuals are positive, others are negative.
- If \hat{u}_i is positive \Rightarrow the line underpredicts y_i
- If \hat{u}_i is negative \Rightarrow the line overpredicts y_i

Definition of
the Simple
Regression
Model

Deriving the
Ordinary Least
Squares
Estimates

Properties of
OLS on any
Sample of
Data

Units of
Measurement
and Functional
Form

Using the Natural
Logarithm in Simple
Regression

Expected
Value of OLS

(1) The sum of the OLS residuals is 0

$$\sum_{i=1}^n \hat{u}_i = 0$$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

(2) The sample covariance between the explanatory variables and the residuals is always zero

$$\sum_{i=1}^n x_i \hat{u}_i = 0$$

- Therefore the sample correlation between the x and \hat{u}_i is also equal to zero.
- Because the \hat{y}_i are linear functions of the x_i , the fitted values and residuals are uncorrelated, too:

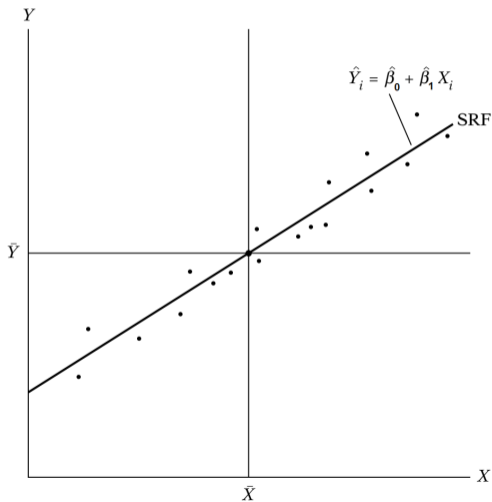
$$\sum_{i=1}^n \hat{y}_i \hat{u}_i = 0$$

(3) The point (\bar{x}, \bar{y}) is always on the OLS regression line.

$$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$$

- That is, if we plug in the average for x , we predict the sample average for y .

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS



Source: Gujarati, Damodar (2002). Basic Econometrics.

Goodness-of-Fit

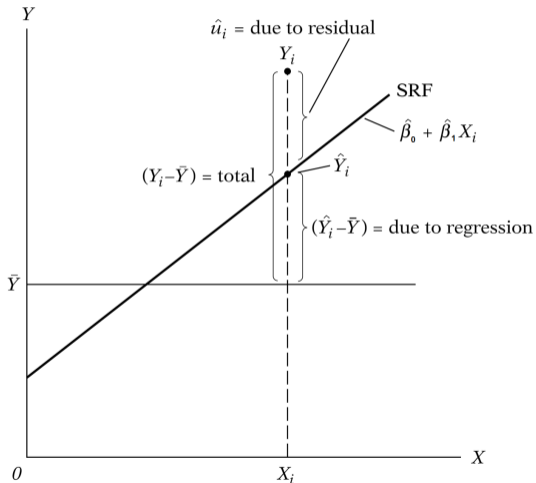
- For each observation, write

$$y_i = \hat{y}_i + \hat{u}_i$$

- Define:

Total Sum of Squares	=	SST	=	$\sum_{i=1}^n (y_i - \bar{y})^2$
Explained Sum of Squares	=	SSE	=	$\sum_{i=1}^n (\hat{y}_i - \bar{y})^2$
Residual Sum of Squares	=	SSR	=	$\sum_{i=1}^n \hat{u}_i^2$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS



Source: Gujarati, Damodar (2002). Basic Econometrics.

(Other names)

- SSR is also know as *Sum of Squared Residuals* or *Model Sum of Residuals*
- $SST = TSS$
- $SSE = ESS$
- $SSR = RSS$

$$\begin{aligned}
 SST &= \sum_{i=1}^n (y_i - \bar{y})^2 \\
 &= \sum_{i=1}^n [(y_i - \hat{y}_i) + (\hat{y}_i - \bar{y})]^2 \\
 &= \sum_{i=1}^n [\hat{u}_i - (\hat{y}_i - \bar{y})]^2
 \end{aligned}$$

Using the fact that the fitted values and residuals are uncorrelated:

$$SST = SSE + SSR$$

The R-Squared

Goal: We want to evaluate how well the independent variable x explains the dependent variable y .

- We want to obtain the fraction of the sample variation in y that is explained by x .
- We will summarize it in one number: R^2 (or **coefficient of determination**.)
- Assuming $SST > 0$,

$$R^2 = \frac{SSE}{SST} = 1 - \frac{SSR}{SST}$$

- Since SSE cannot be greater than the SST , then:

$$0 \leq R^2 \leq 1$$

- $R^2 = 0 \Rightarrow$ **No linear relationship** (between y_i and x_i).
- $R^2 = 1 \Rightarrow$ **Perfect linear relationship** (between y_i and x_i).
- As R^2 increases $\Rightarrow y_i$ gets closer and closer to the OLS regression line.

We should not focus only on R^2 to analyze our regression.

Example (Wage)

$$\widehat{wage}_n = -0.90 + 0.54 educ$$

$$n = 526, \quad R^2 = .16$$

- Therefore, years of education explains only about 16% of the variation in hourly wage.


```

> summary(regression_wage1)

Call:
lm(formula = wage ~ educ, data = wage1)

Residuals:
    Min       1Q   Median       3Q      Max
-5.3396 -2.1501 -0.9674  1.1921 16.6085

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.90485    0.68497  -1.321   0.187
educ         0.54136    0.05325  10.167 <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.378 on 524 degrees of freedom
Multiple R-squared:  0.1648,    Adjusted R-squared:  0.1632
F-statistic: 103.4 on 1 and 524 DF,  p-value: < 2.2e-16

```

Goodness-of-Fit - R output (using *stargazer*)

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

```
> stargazer(regression_wage1, type='text', align=TRUE, digits=2)
```

```
=====
                        Dependent variable:
-----
                        wage
-----
educ                    0.54***
                        (0.05)

Constant                -0.90
                        (0.68)

-----
Observations            526
R2                      0.16
Adjusted R2             0.16
Residual Std. Error    3.38 (df = 524)
F Statistic             103.36*** (df = 1; 524)
=====
Note:                    *p<0.1; **p<0.05; ***p<0.01
```

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Suggested Exercise

Below you have a random sample with 10 data points. Your observations are (x_i, y_i) . Find the $\hat{\beta}_0$, $\hat{\beta}_1$ and the R^2 .

Obs. #	x_i	y_i	x_i	$(y_i - \bar{y})$	$(x_i - \bar{x})$	$(y_i - \bar{y})^2$	$(x_i - \bar{x})^2$	$(x_i - \bar{x})(y_i - \bar{y})$	\hat{y}_i	$(y_i - \hat{y}_i)$	$(\hat{y}_i - \bar{y})^2$	$(y_i - \hat{y}_i)^2$
1	x_1	70	80	-41	-90	1681	8100	3690	65.18	4.82	2099.31	23.21
2	x_2	65	100	-46	-70	2116	4900	3220	75.36	-10.36	1269.95	107.40
3	x_3	90	120	-21	-50	441	2500	1050	85.55	4.45	647.93	19.84
4	x_4	95	140	-16	-30	256	900	480	95.73	-0.73	233.26	0.53
5	x_5	110	160	-1	-10	1	100	10	105.91	4.09	25.92	16.74
6	x_6	115	180	4	10	16	100	40	116.09	-1.09	25.92	1.19
7	x_7	120	200	9	30	81	900	270	126.27	-6.27	233.26	39.35
8	x_8	140	220	29	50	841	2500	1450	136.45	3.55	647.93	12.57
9	x_9	155	240	44	70	1936	4900	3080	146.64	8.36	1269.95	69.95
10	x_{10}	150	260	39	90	1521	8100	3510	156.82	-6.82	2099.31	46.49
	Sum	1,110	1,700	0.00	0.00	8,890	33,000	16,800	1,110	0.00	8,553	337

Definition of the Simple Regression Model

Deriving the Ordinary Least Squares Estimates

Properties of OLS on any Sample of Data

Units of Measurement and Functional Form

Using the Natural Logarithm in Simple Regression

Expected Value of OLS

- ① Definition of the Simple Regression Model
- ② Deriving the Ordinary Least Squares Estimates
- ③ Properties of OLS on any Sample of Data
- ④ Units of Measurement and Functional Form**
Using the Natural Logarithm in Simple Regression
- ⑤ Expected Value of OLS

Example

salary: Annual CEO's salary in thousands of dollars

roe: Average return on equity (measured in percentage)

$$\widehat{salary}_n = 963.19 + 18.50 roe$$

$$n = 209, R^2 = .01$$

- A one unit increase in the independent variable (i.e. *roe* increases one percent) \Rightarrow increases the predicted salary by 18.501, or **\$18,501**.

- If we measure *roe* as a decimal (rather than a percent), what will happen to the intercept, slope, and R^2 ?

We want:

$$roedec = roe/100$$

- What if we measure salary in dollars (rather than thousands of dollars)? what will happen to the intercept, slope, and R^2 ?

We want:

$$salarydol = 1,000 \cdot salary$$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form**
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Changing Units of Measurement

- If the dependent variable y is multiplied by a constant $c \Rightarrow c \cdot \hat{\beta}_0$ and $c \cdot \hat{\beta}_1$
- If the independent variable x is multiplied by a constant $c \Rightarrow \frac{1}{c} \cdot \hat{\beta}_1$

In general, changing the units of measurement of only the independent variable does not affect the intercept

Example: CEO's salary - Original Regression

$$\widehat{salary}_n = 963.19 + 18.50 roe$$

$$n = 209, R^2 = .01$$

Example: CEO's salary - *roe* as a decimal

The new regression is:

$$\widehat{salary}_n = 963.191 + 1,850.1 roedec$$

$$n = 209, R^2 = .01$$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

```
> roedec<-ceosal1$roe*(1/100)
> regression_ceosal1c <- lm(salary ~ roedec, data = ceosal1)
> stargazer(regression_ceosal1c, type='text', align=TRUE, digits=2)
```

```
=====
                        Dependent variable:
-----
                        salary
-----
roedec                    1,850.12*
                        (1,112.33)

Constant                  963.19***
                        (213.24)

-----
Observations                209
R2                          0.01
Adjusted R2                 0.01
Residual Std. Error        1,366.55 (df = 207)
F Statistic                 2.77* (df = 1; 207)
=====
Note:                       *p<0.1; **p<0.05; ***p<0.01
```

Example: CEO's salary - Original Regression

$$\widehat{salary} = 963.19 + 18.50 roe$$

$$n = 209, R^2 = .01$$

Example: CEO's salary - *salary* in dollars

The new regression is

$$\widehat{salarydol} = 963,191 + 18,501 roe$$

$$n = 209, R^2 = .01$$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form**
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

```
> salarydol<-ceosal1$salary*1000
> regression_ceosal1b <- lm(salarydol ~ roe, data = ceosal1)
> stargazer(regression_ceosal1b, type='text', align=TRUE, digits=2)
```

```
=====
                        Dependent variable:
                        -----
                        salarydol
-----
roe                        18,501.19*
                           (11,123.25)

Constant                    963,191.30***
                           (213,240.30)

-----
Observations                 209
R2                           0.01
Adjusted R2                  0.01
Residual Std. Error   1,366,555.00 (df = 207)
F Statistic              2.77* (df = 1; 207)
=====
Note:                *p<0.1; **p<0.05; ***p<0.01
```

- Recall the **wage** example:

Example (Wage)

$$\widehat{wage}_n = -0.90 + 0.54 educ$$

$$n = 526, \quad R^2 = .16$$

- Now, think about the econometric model and how this OLS Regression Function is interpreted.
- What the OLS Regression Line says may not fit how economically we see the problem.

Possible issue: the dollar value of another year of schooling is constant.

- So the 16th year of education is worth the same as the second.
- We expect additional years of schooling to be worth more, in dollar terms, than previous years.
- How can we incorporate an increasing effect? One way is to postulate a constant *percentage* effect.
- We can approximate percentage changes using the natural log.

Constant Percent Model

- Let the dependent variable be $\log(wage)$ and write a (new) simple linear regression model:

$$\log(wage) = \beta_0 + \beta_1 educ + u$$

- Let's define $\log(wage)$ (write it as $lwage$) and run a new regression.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

```

=====
                                Dependent variable:
                                -----
                                lwage
-----
educ                            0.08***
                                (0.01)

Constant                         0.58***
                                (0.10)

-----
Observations                     526
R2                               0.19
Adjusted R2                      0.18
Residual Std. Error              0.48 (df = 524)
F Statistic                      119.58*** (df = 1; 524)
=====
Note:                            *p<0.1; **p<0.05; ***p<0.01
  
```

$$\widehat{lwage}_n = 0.58 + .08 educ$$

$$n = 526, R^2 = .19$$

- The estimated return to each year of education is about 8%.

- **Attention:**

This R -squared is not directly comparable to the R -squared when $wage$ is the dependent variable. The total variation (SSTs) in $wage_i$ and $lwage_i$ that we must explain are completely different.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Constant Elasticity Model

- We can use the log on both sides of the equation to get **constant elasticity models**. For example, if

$$\log(\textit{salary}) = \beta_0 + \beta_1 \log(\textit{sales}) + u$$

then

$$\beta_1 \approx \frac{\% \Delta \textit{salary}}{\% \Delta \textit{sales}}$$

- The elasticity is free of units of *salary* and *sales*.
- A constant elasticity model for salary and sales makes more sense than a constant dollar effect.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Model	Dependent Variable	Independent Variable	Interpretation of β_1
Level-Level	y	x	$\Delta y = \beta_1 \Delta x$
Level-Log	y	$\log(x)$	$\Delta y = (\beta_1/100)\% \Delta x$
Log-Level	$\log(y)$	x	$\% \Delta y = (100\beta_1) \Delta x$
Log-Log	$\log(y)$	$\log(x)$	$\% \Delta y = \beta_1 \% \Delta x$

Using the Natural Logarithm in Simple Regression

Definition of the Simple Regression Model

Deriving the Ordinary Least Squares Estimates

Properties of OLS on any Sample of Data

Units of Measurement and Functional Form

Using the Natural Logarithm in Simple Regression

Expected Value of OLS

- Recall the **CEO salary** example, but now the independent variable is *sales*.

$$salary = \beta_0 + \beta_1 sales + u$$

- Applying log on both variables (dependent and independent) we get:

Example (CEO salary)

$$\begin{aligned} \widehat{\log(salary)} &= 4.82 + 0.26 \log(sales) \\ n &= 209, \quad R^2 = .21 \end{aligned}$$

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

```

=====
                                Dependent variable:
                                -----
                                log(salary)
-----
log(sales)                       0.26***
                                   (0.03)

Constant                          4.82***
                                   (0.29)

-----
Observations                       209
R2                                  0.21
Adjusted R2                         0.21
Residual Std. Error                0.50 (df = 207)
F Statistic                        55.30*** (df = 1; 207)
=====
Note:                               *p<0.1; **p<0.05; ***p<0.01
    
```

- The estimated elasticity of CEO salary with respect to firms sales is about .26.
- A **10 percent** increase in sales is associated with a

$$.26(10) = 2.6$$

percent increase in salary.

Definition of
the Simple
Regression
Model

① Definition of the Simple Regression Model

Deriving the
Ordinary Least
Squares
Estimates

② Deriving the Ordinary Least Squares Estimates

Properties of
OLS on any
Sample of
Data

③ Properties of OLS on any Sample of Data

Units of
Measurement
and Functional
Form

④ Units of Measurement and Functional Form
Using the Natural Logarithm in Simple Regression

Using the Natural
Logarithm in Simple
Regression

Expected
Value of OLS

⑤ Expected Value of OLS

The 4 Assumptions for Unbiasedness

Definition of the Simple Regression Model

Deriving the Ordinary Least Squares Estimates

Properties of OLS on any Sample of Data

Units of Measurement and Functional Form

Using the Natural Logarithm in Simple Regression

Expected Value of OLS

Goal: We want to study statistical properties of the OLS estimator

- In order to that, we will need to impose 4 assumptions.

Assumption SLR.1 (Linear in Parameters)

The population model can be written as

$$y = \beta_0 + \beta_1 x + u$$

where β_0 and β_1 are the (unknown) population parameters.

- What linear in parameters mean?
- **Example of non linear in parameters on white board**

Definition of the Simple Regression Model

Deriving the Ordinary Least Squares Estimates

Properties of OLS on any Sample of Data

Units of Measurement and Functional Form

Using the Natural Logarithm in Simple Regression

Expected Value of OLS

Assumption SLR.2 (Random Sampling)

We have a **random sample** of size n , $\{(x_i, y_i) : i = 1, \dots, n\}$, following the population model.

Assumption SLR.3 (Sample Variation in the Explanatory Variable)

The sample outcomes on x_i are not all the same value.

- This is the same as saying the sample variance of $\{x_i : i = 1, \dots, n\}$ is **not zero**.
- If in the population x does not change then we are not asking an interesting question.

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Assumption SLR.4 (Zero Conditional Mean)

In the population, the error term has zero mean given any value of the explanatory variable:

$$E(u|x) = 0 \text{ for all } x.$$

- **Key assumption.**

- We can compute the OLS estimates whether or not this assumption holds.

The 4 Assumptions for Unbiasedness

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

Goal: We want to know if $\hat{\beta}_1$ is unbiased for β_1 , and $\hat{\beta}_0$ is unbiased for β_0

- If,

$$E(\hat{\beta}_1) = \beta_1$$

$$E(\hat{\beta}_0) = \beta_0$$

Then, the **OLS estimator** is unbiased.

- **Demonstration:** On the white board.

Theorem: Unbiasedness of OLS

Under Assumptions SLR.1 through SLR.4

$$E(\hat{\beta}_0) = \beta_0 \text{ and } E(\hat{\beta}_1) = \beta_1 ,$$

for any values of β_0 and β_1 , i.e., $\hat{\beta}_0$ is unbiased for β_0 , and $\hat{\beta}_1$ is unbiased for β_1

- Definition of the Simple Regression Model
- Deriving the Ordinary Least Squares Estimates
- Properties of OLS on any Sample of Data
- Units of Measurement and Functional Form
- Using the Natural Logarithm in Simple Regression
- Expected Value of OLS

- Therefore, the four assumptions for the OLS estimator to be unbiased are:

SLR.1: (Linear in Parameters) $y = \beta_0 + \beta_1x + u$

SLR.2: (Random Sampling)

SLR.3: (Sample Variation in x_i)

SLR.4: (Zero Conditional Mean) $E(u|x) = 0$

- If any of these assumptions fails, the OLS estimator will (generally) be biased.
- **To be discussed in the next chapter:** What are the omitted factors? Are they likely to be correlated with x ? If so, SLR.4 fails and OLS will be biased.